Chapter IV

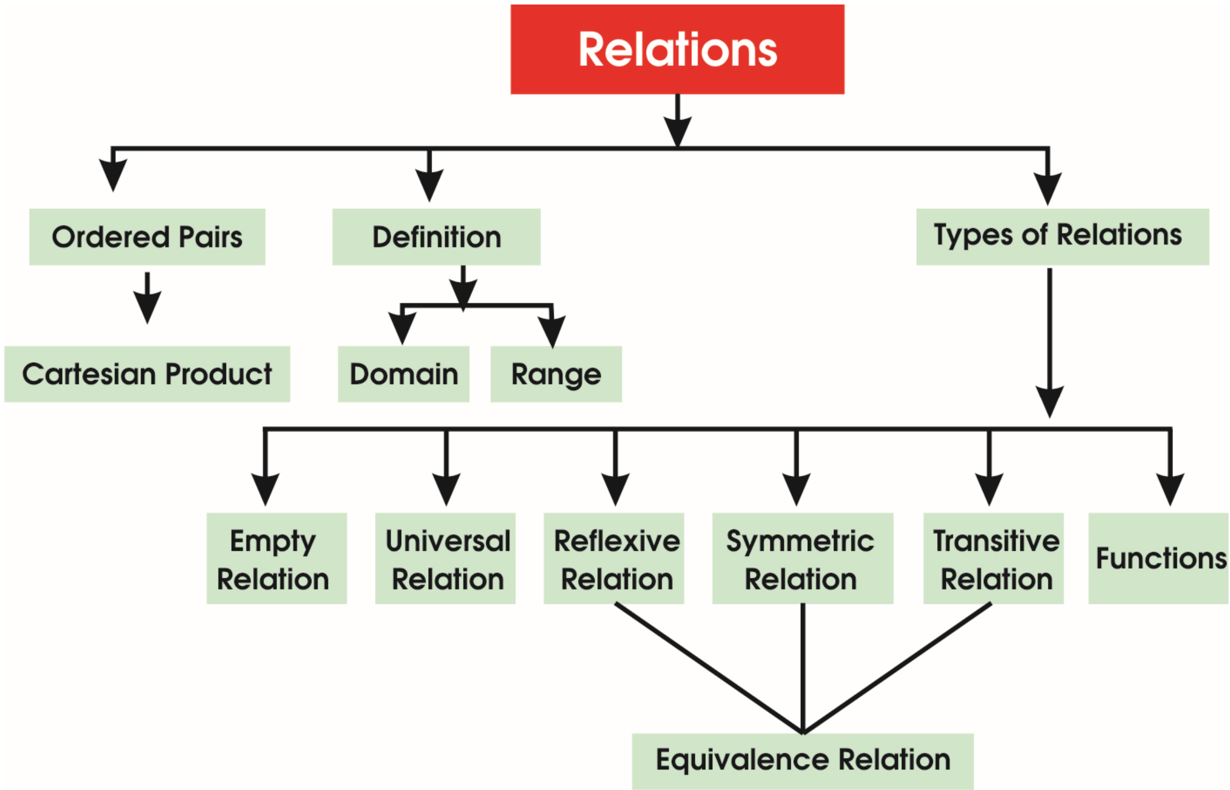
Relations

Learning Objectives:

After the completion of this section, the students will be able to:

- Define an ordered pair.
- Understand the concept of cartesian product of sets.
- Relate relations as a subset of cartesian product.
- Identify different types of relations.

Concept Map



4.1 Introduction

A relation is a way in which two or more people or things are connected. In day to day life, we come across many such relations, such as, brother and sister, mother and daughter, teacher and student, table and its cost price and the list is endless. In mathematics too, we come across many relations, for e.g., number 'y' is twice the number 'x', set A is a subset of set B, a line l_1 is parallel to line l_2, volume of a sphere with its radius r is $\frac{4}{3}\pi r^3$ and so on. In all these examples we observe that a relation involves pairs of objects in a certain order.

In this chapter, we will understand how to connect pairs of objects from two sets and then define a relation between two objects of the pair. After having understood the concept of relations, we would do different types of relations.

For understanding 'relation', we first need to know about ordered pairs and Cartesian product of sets.

4.2 Ordered Pair

What is an ordered pair?

Definition 1: An ordered pair is a pair of objects taken is a specific order and the order in which they appear in the pair is significant. The ordered pair is written as (a, b), where 'a' is the first member of the pair and 'b' is the second member of the pair.

For example, if we form pairs of stationery items along with its price, we may write it as (pencil, Rs. 1), (pen, Rs. 10), (eraser, Rs. 5) and so on. It is evident from these examples, that the first member represents the stationery item and second member, its cost per piece.

Definition 2: Equality of ordered pairs:

Two ordered pairs (a, b) and (c, d) are said to be equal if a = c and b = d, that is, to say,

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d$$

Note:

1. The order in which the elements of the pair occur should be meaningful.
2. The ordered pairs (a, b) and (b, a) are different, unless a = b.
3. Neither (a, b) represents an interval nor does it represent a set {a, b}.

Illustration 1: If the ordered pairs $(2x, y - 3)$ and $(2, 1)$ are equal, then find x and y.

Solution: Since, $(2x, y - 3) = (2, 1)$, we have

$$2x = 2 \text{ and } y - 3 = 1$$

$$\Rightarrow \quad x = 1 \text{ and } y = 4$$

Illustration 2: Find 'x' and 'y' if $(x^2 - 4x, y^2 - y) = (-4, 6)$

Solution: Since, $(x^2 - 4x, y^2 - y) = (-4, 6)$

$\Rightarrow \quad x^2 - 4x = -4$ and $y^2 - y = 6$

$\Rightarrow \quad x^2 - 4x + 4 = 0$ and $y^2 - y - 6 = 0$

$\Rightarrow \quad (x - 2)^2 = 0$ and $(y - 3)(y + 2) = 0$

$\Rightarrow \quad x = 2$ and $y = -2, 3$

4.3 Cartesian Product of Two Sets

Let P and Q be two non-empty sets. Then the Cartesian product of P and Q in this order is written as P x Q and is defined as the set of all ordered pairs (p,q) such that $p \in P$, $q \in Q$ that is,

$$P \times Q = \{(p, q) : p \in P, q \in Q\}$$

Illustration 3: Let A = {a, b, c}, B = {1, 2}

Find A x B and B x A. Is A x B = B x A?

Solution: Given A = {a, b, c} and B = {1, 2}, then by definition of Cartesian product, we have,

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

$$B \times A = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}$$

Here, A x B and B x A do not have exactly the same elements (ordered pairs) $\therefore$ $A \times B \neq B \times A$

We conclude that, in general, $A \times B \neq B \times A$.

In case A x B = B x A, then A = B

Note:

1. In general $A \times B \neq B \times A$. In case A x B = B x A, then A = B.
2. $A \times B = \varnothing$ if either A or B or both are empty sets.
3. $A \times B \neq \varnothing$ if both A and B are non-empty sets.
4. $n(A \times B) = n(A) \times n(B)$, where n (A) denotes the number of elements in the set A, that is,

if n (A) = p and n (B) = q, then

n (A x B) = pq

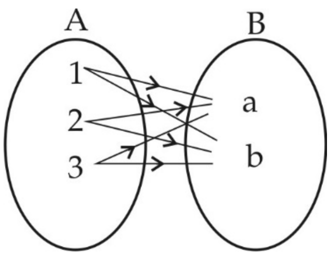
5. A x B x C = {(a, b, c) : a $\in$ A, b $\in$ B, c $\in$ C}

   Here, (a, b, c) is called an ordered triplet and n(A x B x C) = n(A) x n(B) x n(C)

6. If A and B are non-empty sets and either A or B is infinite then (A x B) is also an infinite set.

Let us understand Cartesian product of two sets, say, A and B, through arrow diagrams.

Let A = {1, 2, 3} and B = {a, b}. Then n(A x B) = 3 x 2 = 6



A x B = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}

Illustration 4: Let A = {2, 3} and B = {4, 5}. Find:

1. A x B
2. B x A
3. n (A x B)
4. number of subsets of A x B

Solution: Now, A = {2, 3}, B = {4, 5}

1. A x B = {(2, 4), (2, 5), (3, 4), (3, 5)}
2. B x A = {(4, 2), (4, 3), (5, 2), (5, 3)}
3. n (A x B) = n (A) x n(B) = 2 x 2 = 4
4. Now n(A x B) = 4

$\therefore$ Number of subsets of A x B are $2^4 = 16$

Note: A x B and B x A are equivalent sets as the n(A x B) = n(B x A)

Illustration 5: If A x B = {(a, x), (a, y), (b, x), (b, y)}, then find A and B.

Solution: A x B = {(a, x), (a, y), (b, x), (b, y)}

Then, A = {a, b} and B = {x, y}

Illustration 6: If (–1, 1), (2, 3), (1, 0), (2, 1) are some of the elements of A x B, then find A and B. Also find the remaining elements of A x B.

Solution: Given (–1, 1), (2, 3), (1, 0), (2, 1) $\in$ A B

$\Rightarrow$ A = {–1, 1, 2}, B = {0, 1, 3}

$\therefore$ n (A x B) = 9

Therefore, there are 5 remaining elements which are:

(–1, 3), (–1, 0), (2, 0), (1, 1) and (1, 3)

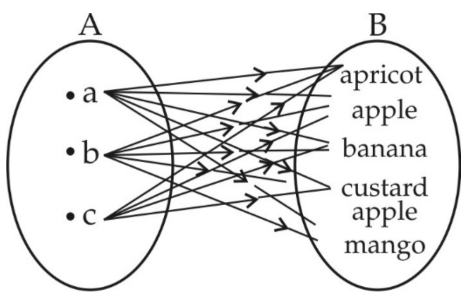
4.4 Relations

Consider two non-empty sets, say, A = {a, b, c} and set B = {apple, apricot, banana, custard apple, mango}.

Now, the Cartesian product of A and B, i.e., A x B has 15 elements, listed as

A x B = {(a, apple), (a, apricot), (a, banana)..... (c, mango)}

Refer to Fig. 1 to depict A x B using arrow diagram



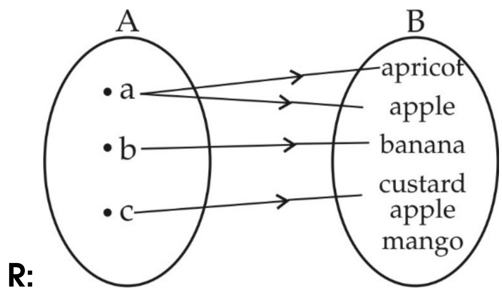
We can now obtain a subset of A x B by defining a relation R between the first element belonging to A and second belonging to B of each ordered pair (a, b) as:

R = {(a, b) : a is the first letter of the name of the fruit in B, a $\in$ A, b $\in$ B}

Therefore, R = {(a, apple), (a, apricot), (b, banana), (c, custard apple)}

Here, R is said to be a 'Relation from the set A to the set B'

An arrow diagram of this relation R is as shown in Figure 2.



Note: A Relation 'R' from a set A to a set B is a subset of A x B.

Definition 3: A relation 'R' from a non-empty set A to a non-empty set B is defined to be a subset of Cartesian product A x B. If (a,b) $\in$ R, then the second elements is called the image of the first element a. Further, we say that a is R - related to b and we write a R b.

Definition 4: The set of all the first elements of the ordered pairs in a relation 'R' from set A to a set B is called the domain of the relation R.

Definition 5: The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The entire set B is called the co-domain of the relation R.

Note: Range $\subseteq$ co-domain

Illustration 6: Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by:

R = {(x, y) : x > y}

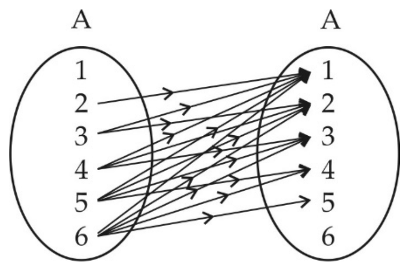
Note : A Relation from A to itself is called a relation defined on A (or simply, a relation in A)

1. Write the given relation in tabular or roster form.
2. Define this relation using an arrow diagram.
3. Write the domain, range and co-domain of R.

Solution:

1. R = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)}

2.



3. Domain = {2, 3, 4, 5, 6}

   Range = {1, 2, 3, 4, 5}

Co-domain = A={1,2,3,4,5,6}

Illustration 7: Let A be a non-empty set with n(A) = 3. Find the number of relations from set A to itself.

Solution: By the definition, a relation from any set A to set B is a subset of A x B.

Now, n(A x A) = n(A) n(A) = 9

$\therefore$ Number of subsets of A x A are 2^9.

Note: Let R be a relation from set A to set B, where n(A) = p and n(B) = q.

Then the total number of relations from A to B are given by 2^{pq} as relations are subsets of A x B and number of subsets of the set A x B containing pq elements are 2^{pq}.

4.5 Types of Relations

In this section we would be discussing different types of relations. In this section, we will also be dealing with functions from set A to itself.

Definition 6: Empty Relation

A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

Illustration 8: Let A be the set of all natural numbers. Define $R = \{(x, y); \frac{x}{y} < 0; x, y \in A\}$.

Solution: Since all natural numbers are always greater than zero, therefore, division of no two natural numbers will give a negative real number. This shows that relation R on A x A is an empty relation.

Definition 7: Universal Relation

A relation R in a set A is called universal relation, if each element of A is related to every element of A, i.e., R = A x A.

Illustration 9: Let A = {1, 2, 3, 4, 5}

Define $R = \{(x, y) : x + y \in N; x, y \in A\}$

Solution: $R = \{(x, y) : x + y \in N; x, y \in A\}$

= {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)}

= A x A

Hence, R is a universal relation

Equivalence relations

Let us introduce equivalence relation, which is one of the most important relations and plays a very significant role. In order to study equivalence relation we first understand the three types of relations, i.e., reflexive, symmetric and transitive.

Definition: A relation R in a set A is called

1. Reflexive, if $(a, a) \in R$, for every $a \in A$
2. Symmetric, if $(a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$
3. Transitive, if $(a, b) \in R$ and $(b, c) \in R$

   $\Rightarrow (a, c) \in R$, for all $a, b, c \in A$

Definition: A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

Example 1: Let T be the set of all triangles in a plane with R a relation in T given by

$R = \{(T_1, T_2) : (T_1 \sim T_2\}$. Show that R is an equivalence relation.

Solution: *R is reflexive Let* $T_1 \in T$

Since every triangle is similar to itself.

i.e, $T_1 \sim T_1 \quad \forall\, T_1 \in T$

This means that $(T_1, T_1) \in T \ \forall\ T_1 \in T$

Therefore, R is reflexive relation

R is symmetric Let $T_1, T_2 \in T$ such that $T_1 \sim T_2$

Then $T_2 \sim T_1$ we conclude R is a symmetric relation.

R is transitive Let $T_1, T_2, T_3 \in T$ such that $(T_1, T_2), (T_2, T_3) \in T$

Then $T_1 \sim T_2$ and $T_2 \sim T_3 \Rightarrow T_1 \sim T_3 \Rightarrow (T_1, T_3) \in T$

Hence R is also transitive

Since R is reflexive, symmetric and transitive. Therefore R is an equivalence relation.

Example 2: Let S be any non empty set and R be a relation defined on power set of S i.e. on P(S) by A R B iff $A \subset B$ for all $A, B \in P(S)$. Show that R is reflexive and transitive but not symmetric.

Solution: R is reflexive

For every $A \in P(S)$, $A \subset A$ i.e., A R A

R is not symmetric

We have $\varnothing, A \in P(s)$ where A contains at least one element. $\varnothing \subset A$, But $A \not\subset \varnothing$

Hence R is not symmetric

R is transitive

For $A, B, C \in P(S)$, if $A \subset B$ and $B \subset C \Rightarrow A \subset C$

Thus for, $(A, B), (B, C) \in R \Rightarrow (A, C) \in R \;\forall\; A, B, C \in P(S)$

Hence R is transitive

Example 3: Show that the relation R in the set Z of integers given by

$$R = \{(a, b) : 3 \text{ divides } a - b\}$$

is an equivalence relation.

Solution: *R is reflexive*

As 3 divides $(a - a)$ for every $a \in Z$, $\therefore$ $(a, a) \in R \;\forall\; a \in Z$

R is symmetric

If $(a, b) \in R$ then 3 divides $a - b$

i.e. $a - b = 3\lambda$, where λ is any integer

$\Rightarrow b - a = -3\lambda$

then 3 divides $b - a$

$\Rightarrow (b, a) \in R$

Therefore R is symmetric

R is transitive

If $(a, b), (b, c) \in R$

Then 3 divides $(a - b)$ and $(b - c)$

$a - b = 3\lambda$ and $b - c = 3\mu$, where λ and μ are intergers

$a - c = (a - b) + (b - c) = 3\lambda + 3\mu$

$a - c = 3(\lambda+\mu)$

$a - c = 3k \Rightarrow a - c = 3$ (an Integer)

3 divides $a - c \Rightarrow (a, c) \in R$

This shows R is transitive

Thus, R is an equivalence relation in Z.

Exercise

1. Determine whether each of the following relations are reflexive, symmetric and transitive.
   (i) Relation R in a set S = {1, 2, 3, 4, 5} as R = {(x, y) : y is divisible by x}
   (ii) Relation R in a set L of all lines in a plane as $R = \{(L_1, L_2) : L_1 \perp L_2\}$
2. Show that the relation R in the set R of real numbers, defined as $R = \{(a, b) : a < b^2\}$ is neither reflexive nor symmetric nor transitive.
3. Show that the relation R in the set Z of integers given by R = {(a, b) : 2 divides a – b} is an equivalence relation.
4. If X = {1, 2, 3, 4}, give an example on X which is
   (i) reflexive and symmetric but not transitive.
   (ii) Symmetric and transitive but not reflexive.
   (iii) Neither reflexive, nor symmetric but transitive.
5. If R_1 and R_2 are equivalence relations in set A, show that $R_1 \cap R_2$ is also an equivalence relation.

Solution:

1. (i) reflexive, not symmetric, transitive
   (ii) not reflexive, symmetric, not transitive

Functions

After having understood the concept of relations, let's now briefly try to understand special type of relations called functions, which will later be discussed in detail in calculus.

Definition: Let A and B be two non-empty sets. A function 'f' from set A to set B is a special type of relation from A to B which assigns to every element in A a unique image in B. Set A, is called the domain of 'f' and elements of set B which have pre-image in A is called the range of 'f'. Set B is called co-domain of 'f'.

Therefore, Range $\subseteq$ co-domain

The function 'f' from A to B is denoted by f: A $\rightarrow$ B. Consider the following example.

Example: Examine each of the following relations given below and state, giving reasons whether it is a function or not?

(i) $R = \{(2, 1), (3, 2), (4, 1), (5, 2), (3, 3)\}$

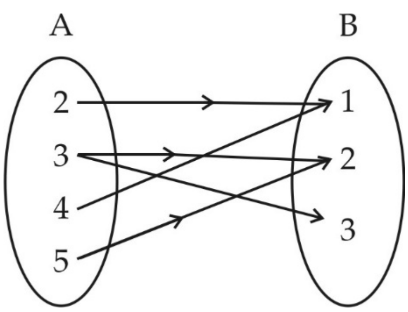
(ii) $R = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$

(iii) $R = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$

(iv) $R = \{(1, 1), (1, 2), (1, 3)\}$

Solution: In this case, domain of R, say, A = {2, 3, 4, 5} and range, say, B = {1, 2, 3}.

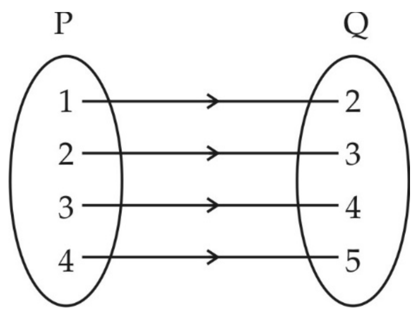
Consider the following arrow diagram from set A to set B.



The given relation is not a function from A to B as there exists an element '3' in the domain which does not have a unique image. (It has two images 2 and 3)

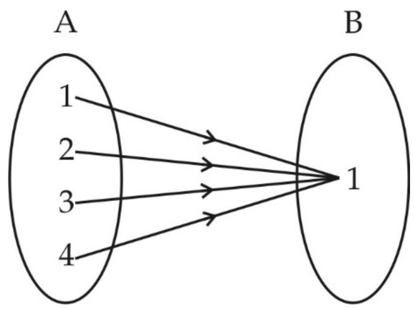
(ii) In this case, domain of R, say, P = {1, 2, 3, 4} and range, say, Q = {2, 3, 4, 5}

According to the arrow diagram, depicting this relation, is given below:



R is a function as every element in P has a unique image in Q.

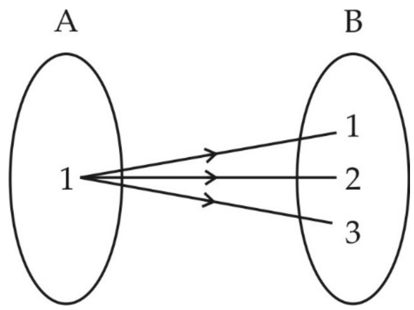
(iii) Consider set A={1,2,3,4} and set B={1} Representing the relation by arrow diagram



Relation R is a function from A to B as every element of set A have unique image in set B. Domain of function is A = {1,2,3,4} and range B = {1}

Consider case

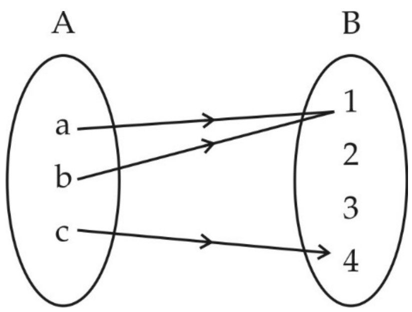
(iv). Here, domain of the relation R, is, say A = {1} and range, say B = {1, 2, 3}. Representing this relation pictorially, we get:



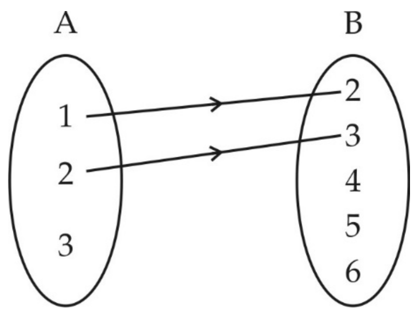
This relation R is not a function from A to B as the element '1' in the domain doesn't have a unique image in B.

Example: Consider the following arrow diagrams depicting relations from set A to set B. Which amongst them are functions? Give reasons.

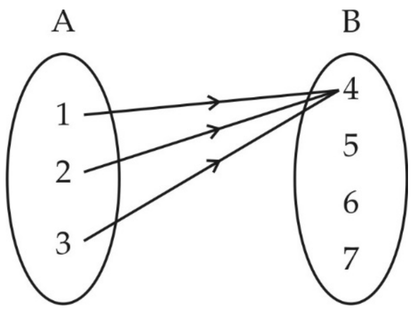
(i)



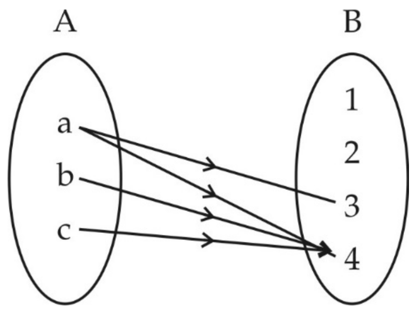
(ii)



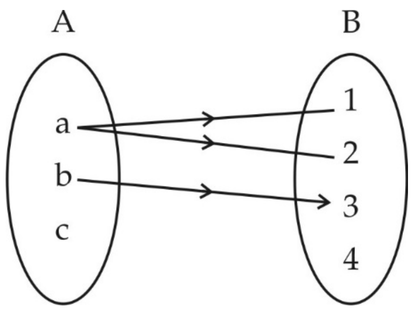
(iii)



(iv)



(v)



Solution: Only Fig. (i) and Fig. (iii) depict functions as every element in set A has a unique image in set B.

In Fig. (ii), the relation is not a function as '3' doesn't have an image in B. Hence, not a function.

In Fig. (iv) element 'a' has two images in set B. Hence, not a function.

In Fig. (v), element 'a' has two images in B and also element 'c' has no image in 'B'. Hence, not a function.

Example: Find the domain and range of each of the following real functions:

(i) $f(x) = x^2$

(ii) $f(x) = \frac{1}{x}$

Solution:

(i) $f(x) = x^2$

Domain: R, set of real numbers

Range: $[0, \infty)$ or non-negative real numbers.

(ii) $f(x) = \frac{1}{x}$

Domain: R – {0}, Range: R – {0}

Summary :

The main features of this section are :

- Ordered pair : A pair of object or elements in a specific order
- Cartesian product of A x B of two sets A and B is given by

  $A \times B = \{(a,b) : a \in A$ and $b \in B$.
- If n(a) = p and n(B) = q then n (A x B) = pq and number of relations possible from A to B are 2^{pq}
- In general, $A \times B \neq B \times A$. In case $A \times B = B \times A$, then A = B.
- Relation 'R' from a non-empty set A to a non-empty set B is a subset of Cartesian product A x B, which is derived by defining a relationship between the first element and the second element of the ordered pair A x B.
- Image is the second element of the ordered pairs in the given relation
- The domain of a relation R is the set of all the first elements of ordered pairs in a relation R.
- The range of a relation R is the set of all second elements of ordered pairs in relation R.
- A Relation R in a set A is called an empty relation if no element of A is related to any element of A.
- A relation R in a set A is called a universal relation if each element of A is related to every element of A
- A relation R on a set A is said to be:

  i) Reflexive if $(a,a) \in R$, for every $a \in A$

  ii) Symmetric if $(a,b) \in R \Rightarrow (b,a) \in R$, where $a, b \in A$

  iii) Transitive if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ where $a,b,c \in A$
- A relation R which is reflexive, symmetric and transitive is called an equivalence relation.